

Synchronization of firefly cellular automata on various graphs

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Joint work with Sam Dittmer and Igor Pak

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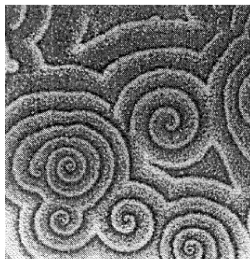
Combinatorial aspects of coupled oscillators

- [1] H. Lyu, “*Global synchronization of pulse-coupled oscillators on trees*” SIAM Journal on Applied Dynamical Systems (to appear) arXiv:1604.08381
- [2] H. Lyu, “*Synchronization of finite-state pulse-coupled oscillators*” Physica D: Nonlinear Phenomena 303 (2015): 28-38. arXiv:1407.1103
- [3] H. Lyu, “*Phase transition in firefly cellular automaton on finite trees.*” arXiv:1610.00837 (2016)

Probabilistic aspects of coupled oscillators

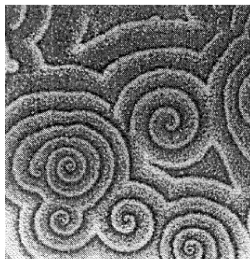
- [4] J. Gravner, H. Lyu, and D. Sivakoff, “*Limiting behavior of 3-color excitable media on arbitrary graphs.*” Annals of Applied Probability (to appear). arXiv:1610.07320
- [5] H. Lyu and D. Sivakoff, “*Persistence of sums of correlated increments and clustering in cellular automata.*” Stochastic Processes and Applications (2018), Vol. 129, Issue 4. arxiv.org/1706.08117
- [6] H. Lyu and D. Sivakoff, “*Synchronization of finite-state pulse-coupled oscillators on \mathbb{Z} .*” arXiv.org:1701.00319 (2017)

Excitable media



- ▶ An **excitable medium** is a network of dynamic units where each unit fluctuates its neighbors' internal dynamics on a particular event

Excitable media



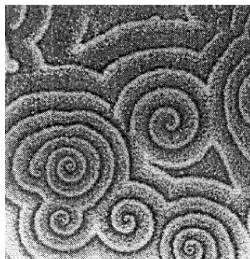
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Figure: (top) Cyclic AMP wave patterns in slime molds (by L. Yang) and (bottom) BZ oscillator (by Abteilung Biophysik Lab)

Simulations: on lattices and its spanning tree



Movie: A4C on square lattice (with Moore neighborhood, deg 8)



Movie: A4C on a uniform spanning tree of the lattice on the left

Pulse-coupled oscillators

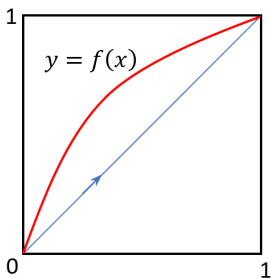


Figure: PRC for an excitatory pulse coupling

- ▶ A **pulse-coupled oscillator** evolves on unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with constant unit speed, fires pulse at phase 1, and adjusts its phase upon receiving pulse from a neighbor.

$$\begin{cases} \dot{\phi}_v(t) \equiv 1 & \text{not upon pulse} \\ \phi_v(t^+) = f(\phi_v(t)) & \text{upon pulse,} \end{cases}$$

- ▶ The way an oscillator responds to pulse signal is given by the **phase response curve** (PRC).
- ▶ PRC is **excitatory** ($f(x) \geq x$), **inhibitory** ($f(x) \leq x$), and **delay-advance**, etc.

Pulse-coupled oscillators

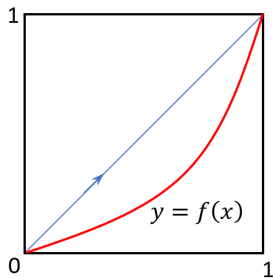


Figure: PRC for an inhibitory pulse coupling

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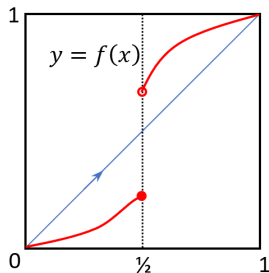


Figure: PRC for a delay-advance pulse coupling

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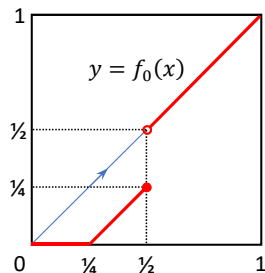
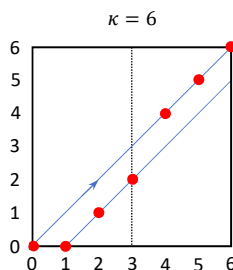
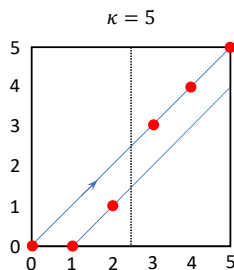
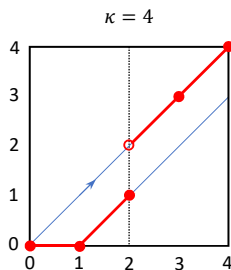


Figure: PRC for the 4-coupling

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- ▶ The way an oscillator responds to pulse signal is given by the **phase response curve** (PRC).
- ▶ The **4-coupling** is the pulse-coupling with the PRC to the left, which extends the 4-color firefly cellular automaton.

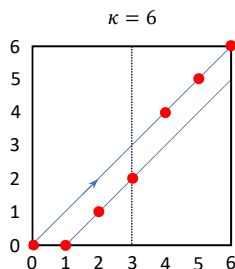
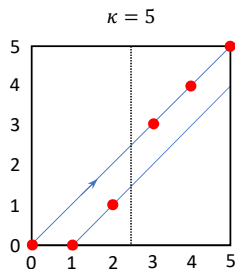
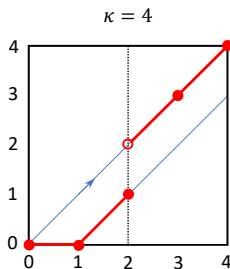
The κ -color Firefly Cellular Automaton (FCA)

- ▶ Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ -coloring $X_t : V \rightarrow \mathbb{Z}_\kappa$ updates in discrete time.

- ▶ The 4-coupling induces the following update rule for $\kappa = 4$:

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) \in \{1, 2\} \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod{4} & \text{otherwise} \end{cases}$$

- ▶ Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

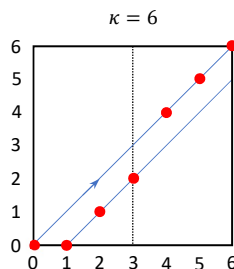
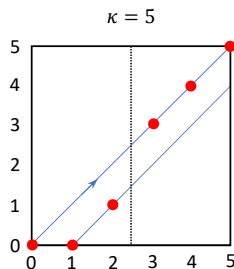
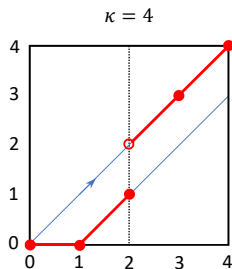
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- ▶ Similar PRC induces the following update rule for $\kappa = 5$:

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) \in \{1, 2\} \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod{5} & \text{otherwise} \end{cases}$$

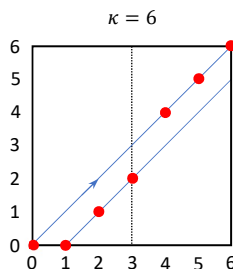
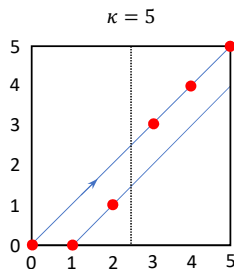
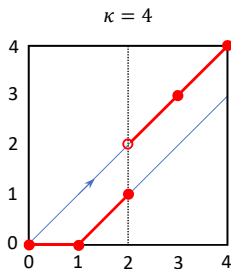
- ▶ Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

The κ -color Firefly Cellular Automaton (FCA)

- ▶ Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ -coloring $X_t : V \rightarrow \mathbb{Z}_\kappa$ updates in discrete time.
- ▶ Similar PRC induces the following update rule for $\kappa = 6$:

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) \in \{1, 2, 3\} \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod{6} & \text{otherwise} \end{cases}$$

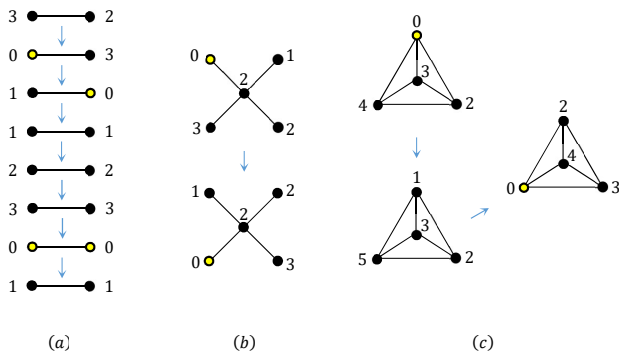
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The κ -color Firefly Cellular Automaton (FCA)

- ▶ Discretize $S^1 = \mathbb{R}/\mathbb{Z}$ into $\mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$, so now a κ -coloring $X_t : V \rightarrow \mathbb{Z}_\kappa$ updates in discrete time.
- ▶ In general, the κ -color FCA $(X_t)_{t \geq 0}$ evolves via

$$X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } 1 \leq X_t(v) \leq \kappa/2 \text{ and} \\ & |\{u \in N(v) : X_t(u) = 0\}| \geq 1 \\ X_t(v) + 1 \pmod{\kappa} & \text{otherwise} \end{cases}$$

- ▶ Say $v \in V$ **blinks** at time t if $X_t(v) = 0$.

Examples of κ -color FCA

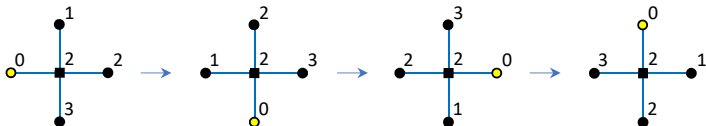
- ▶ 4-color FCA on K_2 : Synchronizes.
- ▶ 4-color FCA on S_4 : Non-synchronizing periodic orbit
- ▶ 6-color FCA on K_4 : Non-synchronizing periodic orbit

Q: Any condition on (G, κ) s.t. every κ -coloring synchronizes?

The κ -color FCA on trees

Theorem (L. 2015, 2017)

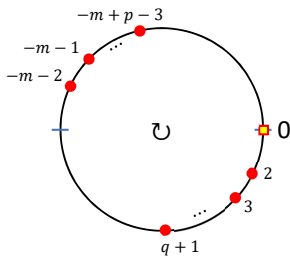
- (i) If $\kappa \in \{3, 4, 5, 6\}$ and $T = (V, E)$ is any finite tree, then every κ -coloring on T synchronizes iff T has maximum degree $< \kappa$.
- (ii) If $\kappa \geq 7$, then there exists a finite tree $T = (V, E)$ with maximum degree $\leq \kappa/2 + 1$ and a non-synchronizing κ -coloring on T .



The κ -color FCA on trees

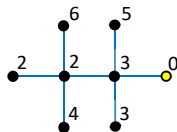
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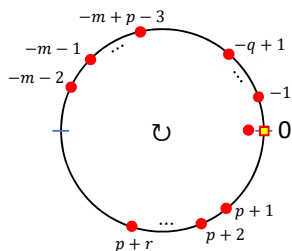


$$\kappa = 2m - 1 \geq 7$$

$$p, q \geq 2, p + q = m$$



$$\kappa = 8$$



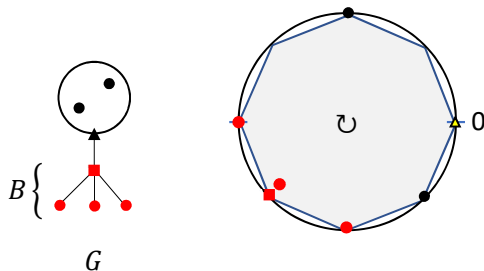
$$\kappa = 2m \geq 10$$

$$p, q, r \geq 2, p + q + r = m + 1$$

Local concentration lemma for FCA

Lemma (Local concentration for FCA)

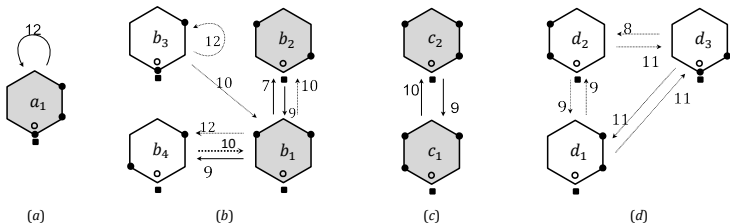
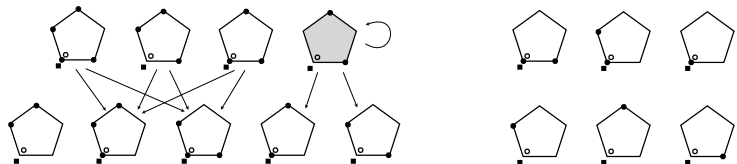
$G = (V, E)$, $B \subseteq G$ a branch, $X_0 : V \rightarrow \mathbb{Z}_\kappa$, $\kappa \geq 3$. If $\omega(X_0|_B) < \kappa/2 - 1$, then the leaves in B become irrelevant of the dynamics after some finite time.



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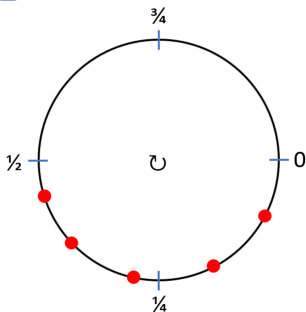
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The concentration (width) lemma

Lemma

$G = (V, E)$ any connected simple graph, $\phi_0 : V \rightarrow S^1$ a phase configuration. If all initial phases are concentrated in an open half of S^1 (i.e., 'width' $\omega(\phi_0) < 1/2$), then ϕ_0 synchronizes under any delay-advance pulse coupling.

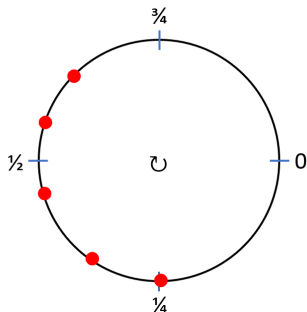


- ▶ The **width** $\omega(\phi_0)$ of ϕ_0 is the length of the shortest arc in S^1 that covers all initial phases in ϕ_0
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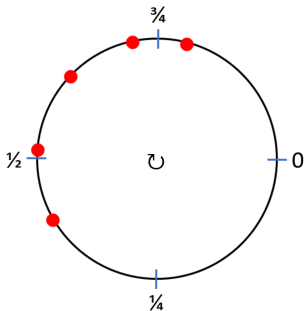


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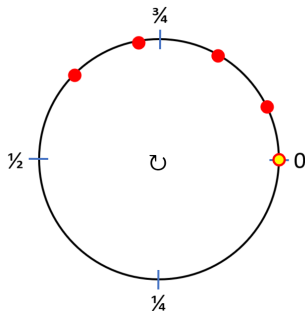


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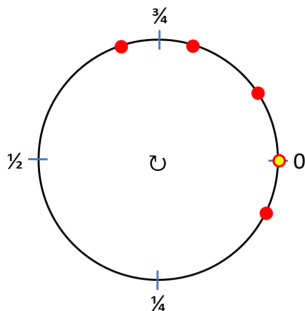


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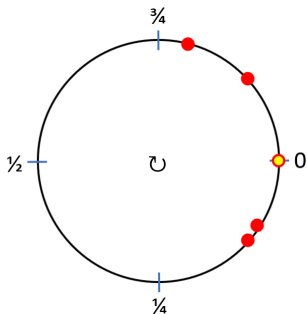


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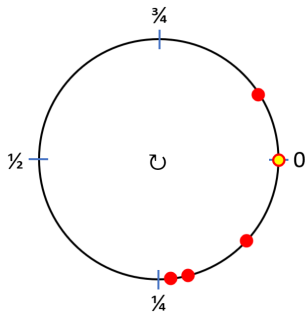


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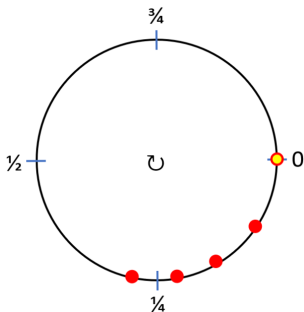


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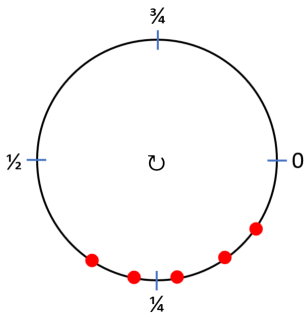


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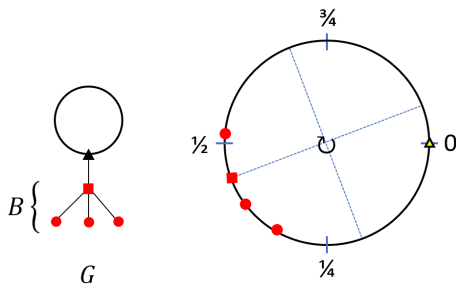
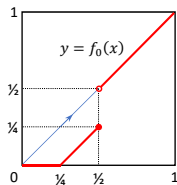
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Lemma

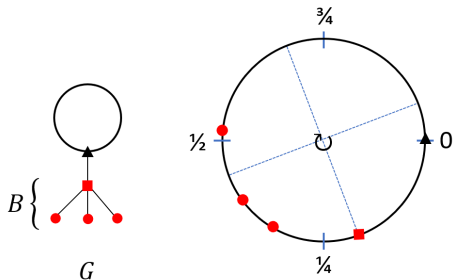
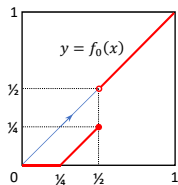
$G = (V, E)$, $B \subseteq G$ a branch, $\phi_0 : V \rightarrow S^1$, $(\phi_t)_{t \geq 0}$ a trajectory evolving under the 4-coupling. If $\omega(\phi_0|_B) < 1/4$, then the leaves in B become irrelevant of the dynamics after some finite time t_0 .



- ▶ The **branch width** $\omega(\phi_0|_B)$ is the width of the restricted phase configuration $\phi_0|_B$ on B .
- ▶ $\omega(\phi_t|_B)$ can only be perturbed by the root \blacktriangle at most once in every 1 second.

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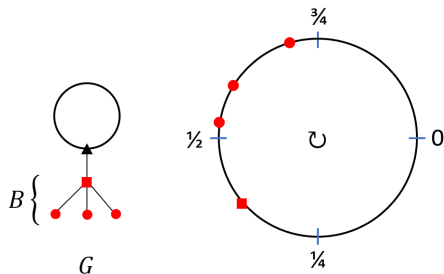
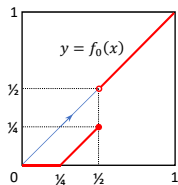
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- ▶ Original branch width could increase by $1/4$
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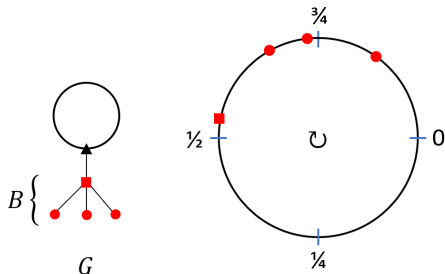
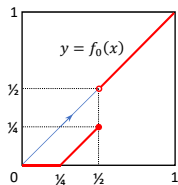
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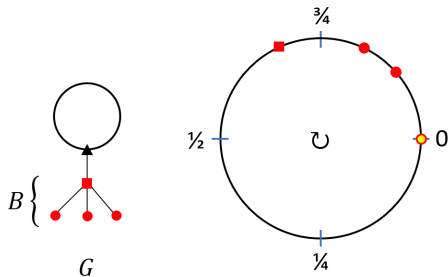
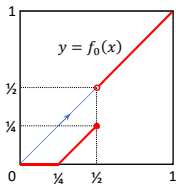
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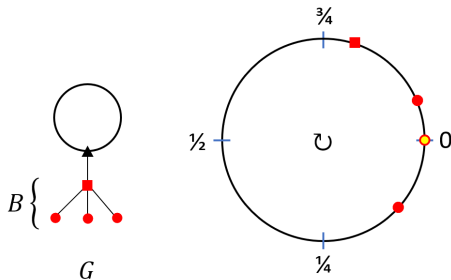
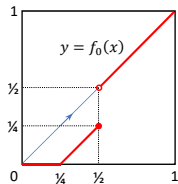
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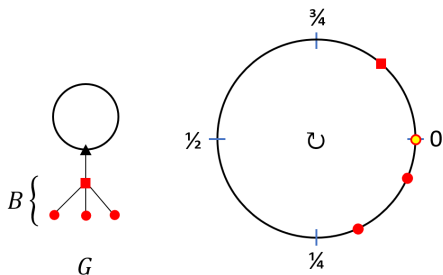
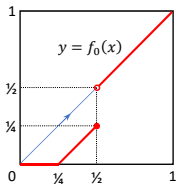
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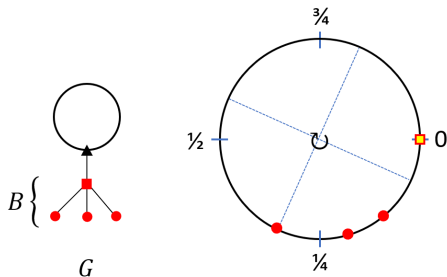
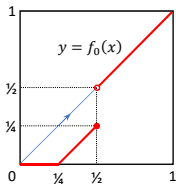
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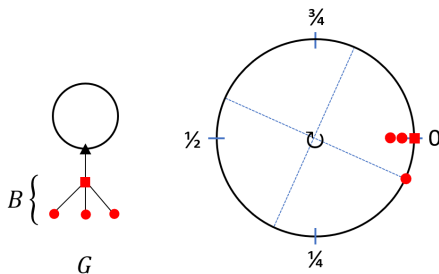
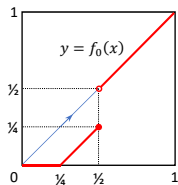
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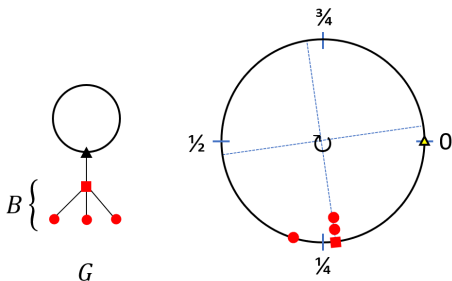
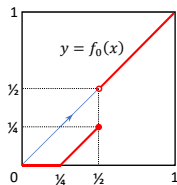
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- ▶ $\omega(\phi_{t+}|_B) \leq \omega(\phi_0|_B)$

Lemma

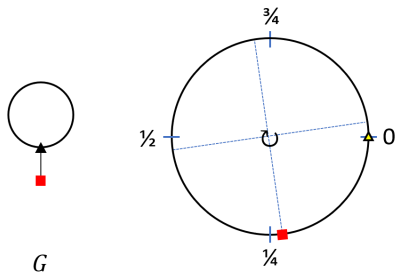
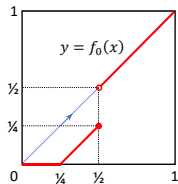
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The 4-coupling on trees

Theorem (L. 2017)

Let $T = (V, E)$ be a finite tree with diameter d . Consider the 4-coupling.

- (i) If T has maximum degree ≤ 3 , arbitrary phase configuration on T synchronizes by time $51d$.
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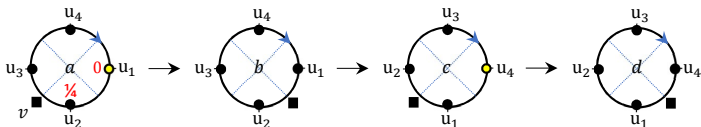
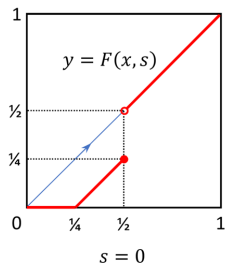
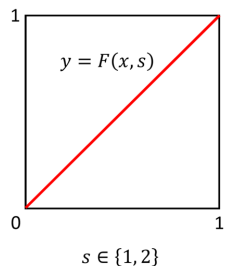


Figure: An example of 4-coupled phase dynamics on a star with center $v = \blacksquare$ and leaves $= \bullet$. In every $1/4$ second, one of the leaves blink and pulls the center by $1/4$ in phase, resulting in a non-synchronizing orbit.

The adaptive 4-coupling



- ▶ In order to overcome the degree constraint, introduce an auxiliary state variable $\sigma_v(t) \in \{0, 1, 2\}$ for each node $v \in V$.
- ▶ Whenever $\sigma_v = 0$, v uses the 4-coupling PRC (top).
- ▶ Whenever $\sigma_v \in \{1, 2\}$, v ignores all input pulses (bottom)
- ▶ Dynamics of this auxiliary variable is carefully coupled with the phase dynamics.



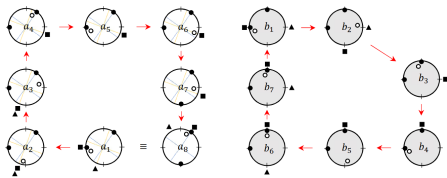
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Lemma

Let $T = (V, E)$ be a finite tree with maximum degree Δ and let $(\Sigma_{\bullet}(t))_{t \geq 0}$ be a joint trajectory on T . Let B be any terminal branch in T . Then $\omega(\phi_{t_1^+} |_B) < 1/4$ for some $t_1 < 157$.



The adaptive 4-coupling on trees

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Corollary (L. 2017)

Consider an autonomous distributed system on an arbitrary finite simple graph $G = (V, E)$ with diameter d and maximum degree Δ . Then $\mathcal{A} = \text{Adaptive 4-coupling} + \text{SpanningTree}$ has the following properties:

- (i) \mathcal{A} can be implemented with $O(\log M\Delta)$ memory per node.
- (ii) Let τ_G be the worst case running time of \mathcal{A} on G . Then $\mathbb{E}[\tau_G] = O(\epsilon M|V| + (d^5 + \Delta^2) \log |V|)$.

Simulations: A4C on a lattice and its spanning tree

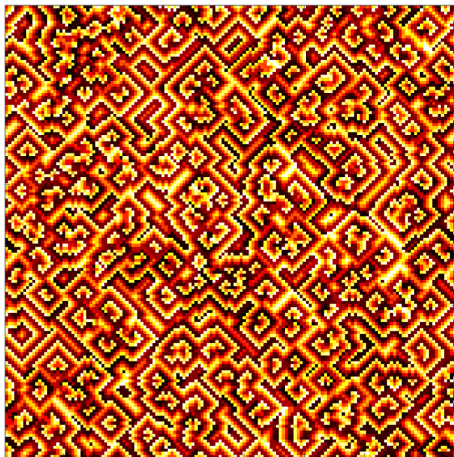


Figure: A4C on square lattice (with Moore neighborhood, deg 8)

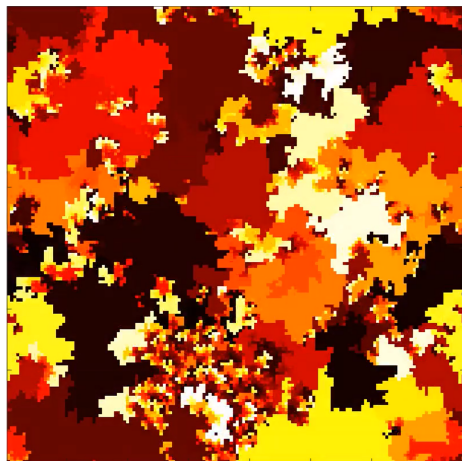


Figure: A4C on a uniform spanning tree of the lattice on the left

Theorem (L. 2015)

P a finite path, $(X_t)_{t \geq 0}$ arbitrary κ -color FCA trajectory on P , $\kappa \geq 3$. Then X_t synchronizes.

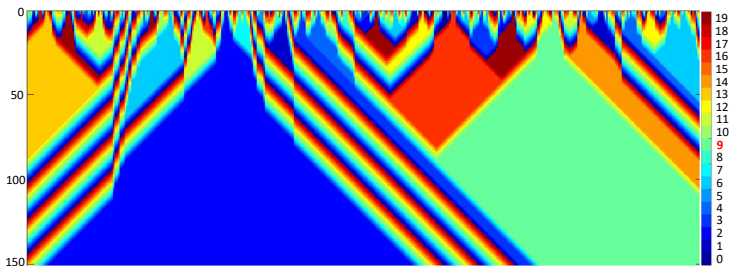


Figure: Simulation of The 20-color FCA on a path of 400 nodes for 150×20 and 70×3 iterations, respectively. The top row shows a random κ -coloring drawn from the uniform product measure, and κ iterations generate each successive row, from top to bottom.

Q. What can we say about the κ -color FCA on the infinite path \mathbb{Z} , started from the uniform product measure?

Let $(X_t)_{t \geq 0}$ be either a 3-color CCA, GHM, or FCA trajectory on \mathbb{Z} , where X_0 is the uniform random 3-coloring of \mathbb{Z} .

Theorem (Fisch 1992)

For the 3-CCA on \mathbb{Z} , we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{2/3\pi t}$.

Theorem (Durrett, Steif 1991)

For the 3-color GHM on \mathbb{Z} , we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{2/27\pi t}$.

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For the 3-color FCA on \mathbb{Z} , we have $\mathbb{P}(X_t(x) \neq X_t(x+1)) \sim \sqrt{8/9\pi t}$.



Figure: Simulations of the 3-color CCA (left), GHM (middle), FCA (right) on \mathbb{Z}

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Figure: Simulations of the 3-color CCA (left), GHM (middle), FCA (right) on \mathbb{Z}

The evolution of “domain walls” behaves like an annihilating particle system:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 2 | 2 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
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Figure: 3-color CCA on \mathbb{Z}



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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | → | 0 | → | 2 | 2 | ← | 0 | → | 2 | ← | 0 | 0 | 0 | → | 2 | ← | 0 | ← | 1 |
| 1 | 1 | → | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ← | 1 | 1 | | | |
| 1 | 1 | 1 | → | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ← | 1 | 1 | 1 | | | | | |
| 1 | 1 | 1 | 1 | → | 0 | 0 | 0 | 0 | ← | 1 | 1 | 1 | 1 | 1 | | | | | |
| 1 | 1 | 1 | 1 | 1 | → | 0 | 0 | ← | 1 | 1 | 1 | 1 | 1 | 1 | | | | | |
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Figure: 3-color CCA on \mathbb{Z}



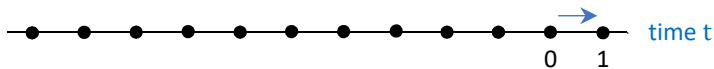
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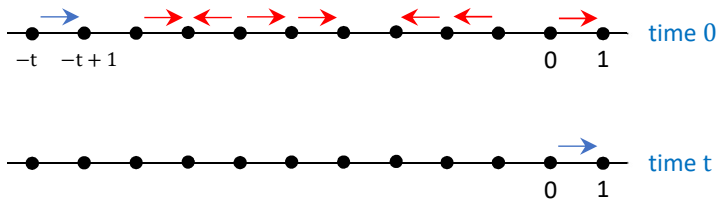
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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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| 1 | 1 | → | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ← | 1 | 1 | 1 |
| 1 | 1 | 1 | → | 0 | 0 | 0 | 0 | 0 | ← | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | → | 0 | 0 | 0 | ← | 1 | 1 | 1 | 1 | 1 |
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| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Figure: 3-color CCA on \mathbb{Z}

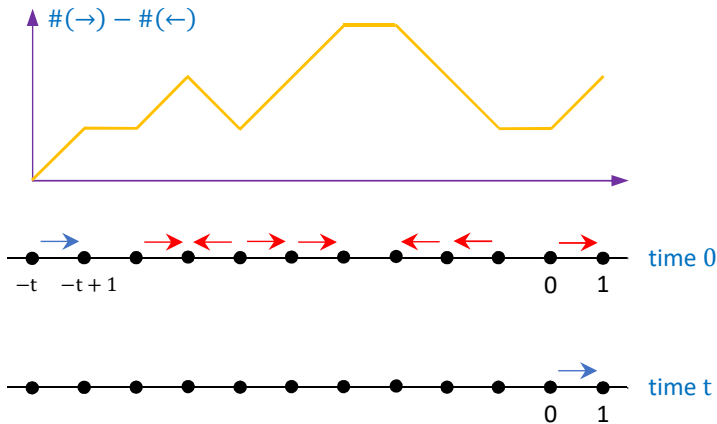
Suppose there is a right particle on the edge $(0, 1)$ at time t .



This particle was distance t away at time 0 and lives up to time t , without being annihilated by a left particle.



This requires $\#(\text{right particle}) > \#(\text{left particle})$ at every intermediate edge.



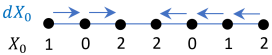
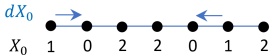
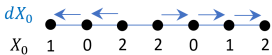
- ▶ Thus we get

$$\begin{aligned} \mathbb{P}\{X_t(0) \neq X_t(1)\} &= 2\mathbb{P}\{dX_t(0,1) = 1\} \\ &= \mathbb{P}\left\{\sum_{x=-t}^s dX_0(x, x+1) \geq 1 \text{ for all } -t \leq s \leq t\right\}. \end{aligned}$$

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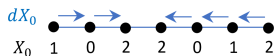
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- ▶ Let $(X_t)_{t \in \mathbb{Z}}$ be a stationary Markov chain on \mathfrak{X} with functional $g : \mathfrak{X} \rightarrow \mathbb{R}$ s.t. $\mathbb{E}(g(X_0)) = 0$ and $\mathbb{E}(g(X_0)^2) < \infty$.
- ▶ Let $S_t = g(X_1) + \dots + g(X_t)$. For each $r \in \mathbb{R}$, and $t \geq 0$, define *survival probabilities* $Q^\bullet(r, t)$ by

$$Q^\bullet(r, t) := \mathbb{P}(S_1 \geq 0, \dots, S_t \geq 0 \mid S_0 = r).$$

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$$\gamma_g^2 := \text{Var}[g(X_0)] + 2 \sum_{k=1}^{\infty} \text{Cov}[g(X_0), g(X_k)].$$

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Theorem (L., Sivakoff, 2017)

If $\gamma_g \in (0, \infty)$ and $(X_{-t})_{t \geq 0}$ is ergodic, then

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Remark: Lemma holds for any stationary process $(X_t)_{t \in \mathbb{Z}}$ without Markov property and second moment condition for $g(X_0)$

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- ▶ So $\mathbb{P}(\tilde{M}(t+1) - \tilde{M}(t) \geq 1) = Q^\bullet(-1, t)$.

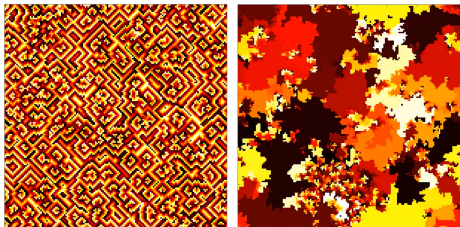


Figure: (Left) A4C on square lattice (with Moore neighborhood, deg 8), (Right) A4C on a uniform spanning tree of the lattice on the left

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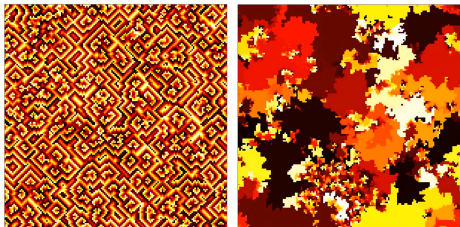


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Question

Is there a (randomized) distributed algorithm \mathcal{T} , which computes a spanning tree T of a given connected graph $G = (V, E)$ with max degree Δ and diameter d , with the following properties?

- \mathcal{T} can be implemented on G with $O(\log \Delta)$ memory per node.
- $\mathbb{E}(\text{worst case running time}) = \text{diam}(G)^{O(1)} \log |V|$.
- $\text{diam}(T) = \text{diam}(G)^{O(1)} \log |V|$.

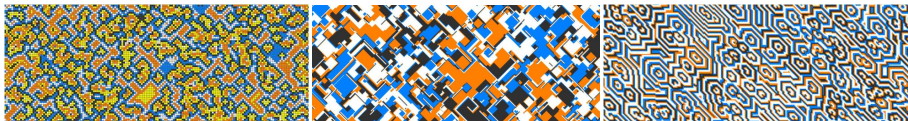


Figure: Snapshots of the 5-color FCA on square lattice (left), 4-color FCA on square lattice (middle), and 4-color FCA on honeycomb lattice (right).

Conjecture

Let $(X_t)_{t \geq 0}$ be the κ -color FCA trajectory on \mathbb{Z}^d started from the uniform product probability measure.

- (i) For $\kappa = 4$ and for all $d \geq 2$, $\lim_{t \rightarrow \infty} \mathbb{P}(X_t(x) = X_t(y)) = 1$ for any $x, y \in \mathbb{Z}^d$.
- (ii) For $\kappa \neq 4$ and for all $d \geq 2$, the trajectory $(X_t)_{t \geq 0}$ converges to a $\kappa + 1$ periodic limit cycle almost surely.

Thank you!