Particle density in diffusion-limited annihilating systems

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Stochastic Analysis on Large Scale Interacting Systems, RIMS

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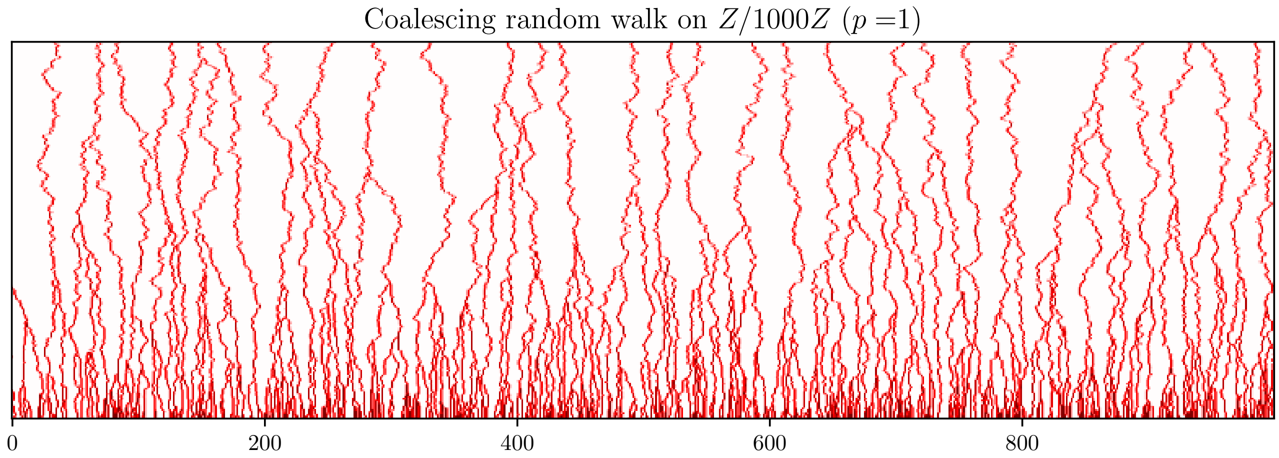
Introduction

Interacting random walks on graphs

- *Coalescing random walk* on a graph G:
  - Initially every vertex has a single particle (of type A)
  - Each particles perform simple random walk at rate 1
  - Collision rule: $A + A \to A$ (no other interaction)



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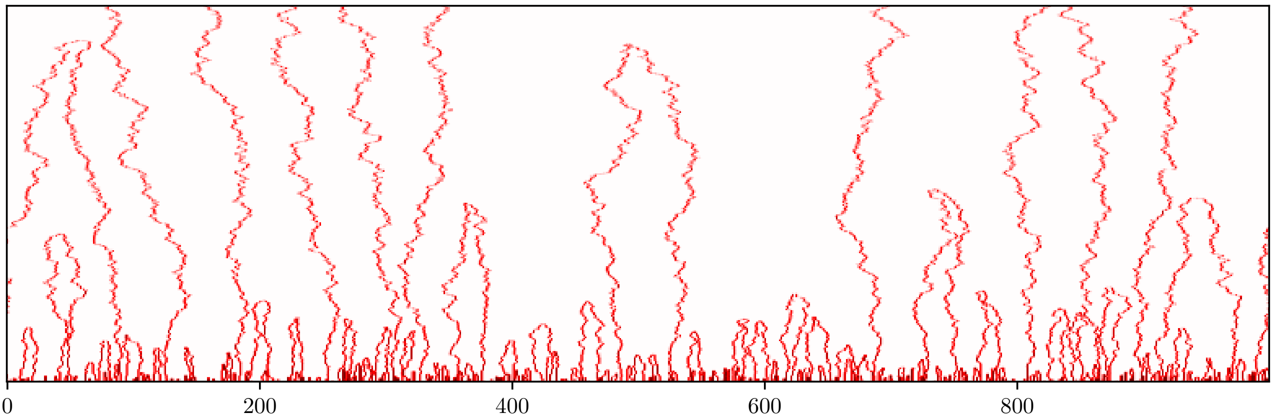
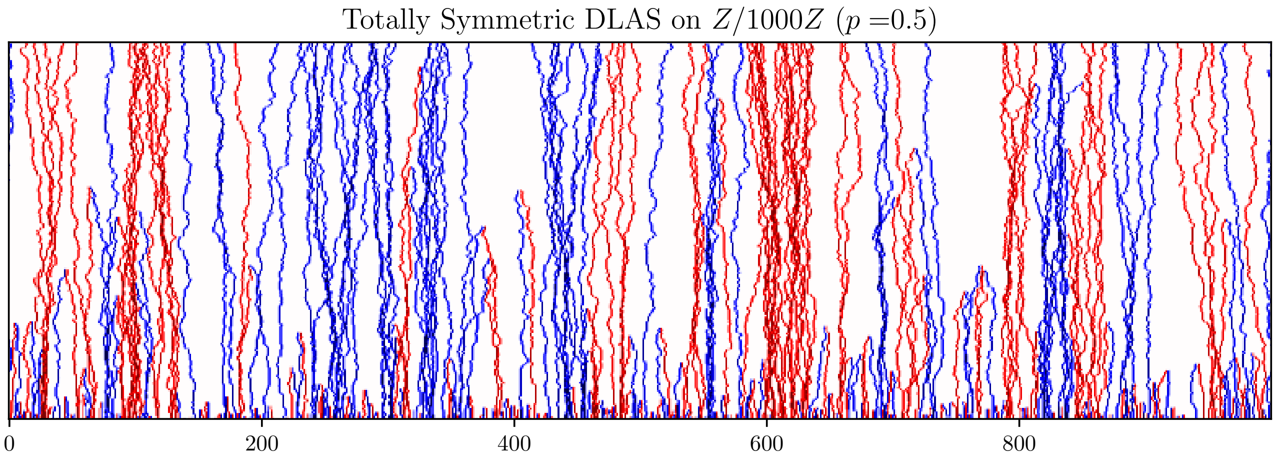
Annihilating random walk on $Z/1000Z$ (p =1)



Interacting random walks on graphs

- *Two-type diffusion limited annihilating systems* (DLAS) on a graph G:
  - Initially every vertex has a particle that is independently of type A with probability $p = p_A$ and otherwise is of type B.
  - A-particles perform simple random walk at rate λ_A and B-particles at rate λ_B.
  - Collision rule: $A + B \to \emptyset$ (no other interaction)
- *Totally Symmetric DLAS*: $p = 0.5$, $\lambda_A = \lambda_B = 1$



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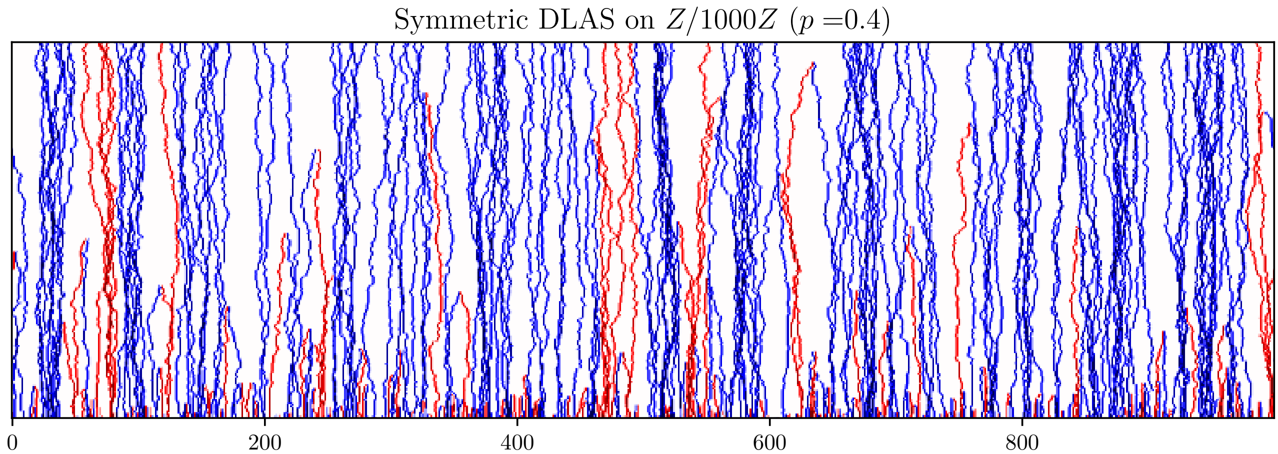
- *Symmetric DLAS with less A-particles*: $p = 0.4$, $\lambda_A = \lambda_B = 1$



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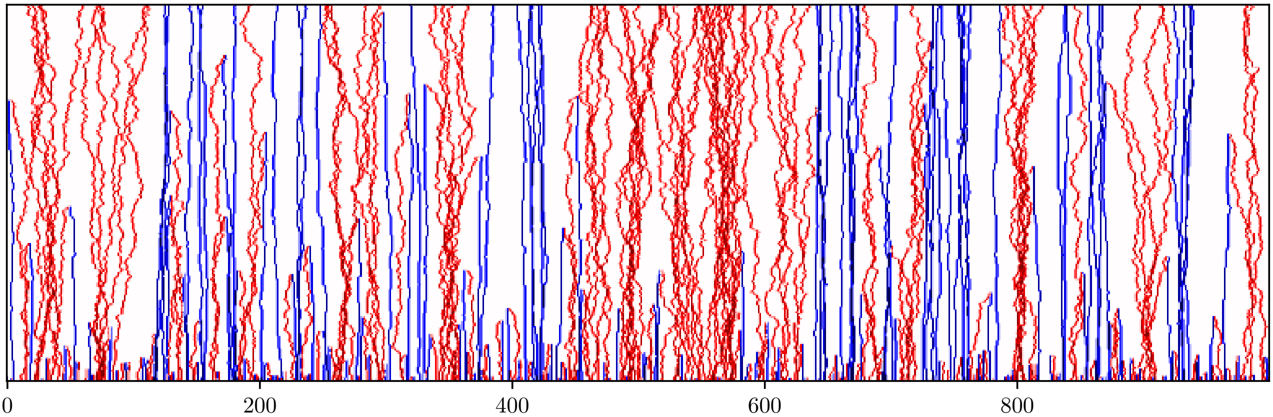
- *Aymmetric DLAS with slow B-particles*: $p = 0.5$, $\lambda_A = 1$, $\lambda_B = 1/10$

Asymmetric DLAS on $Z/1000Z$ (p =0.5)



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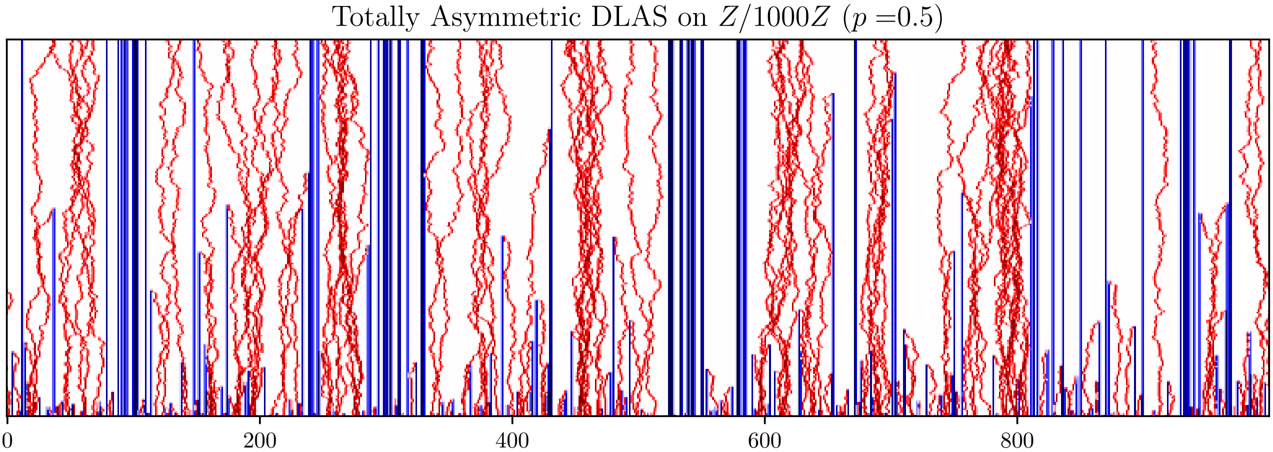
- a.k.a. the *critical parking process* ($p = 0.5$, A-particles = Cars, B-particles = Spots)



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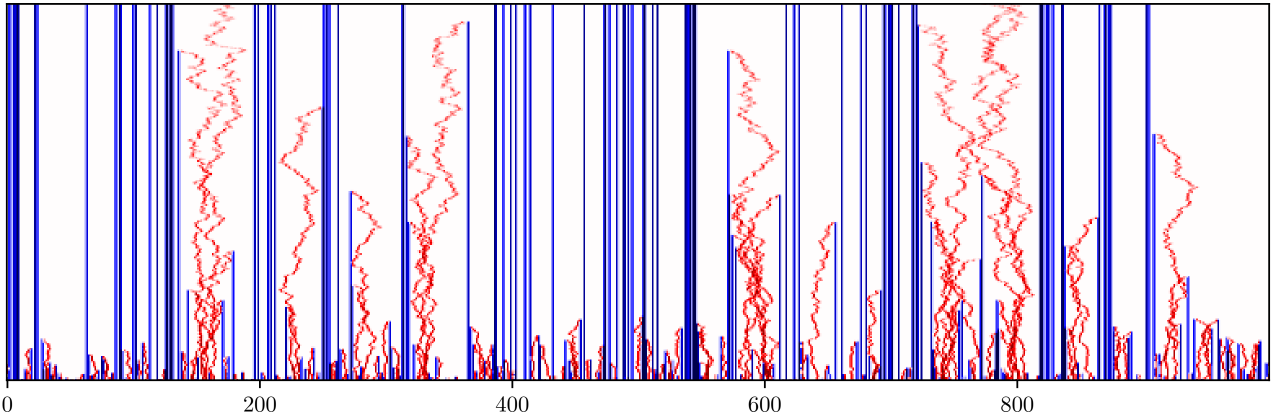

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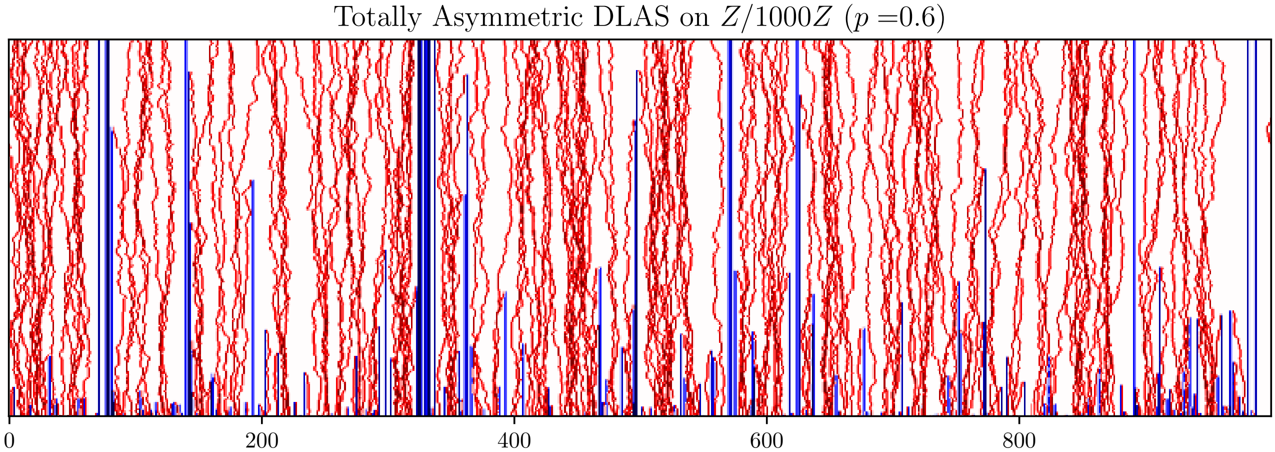
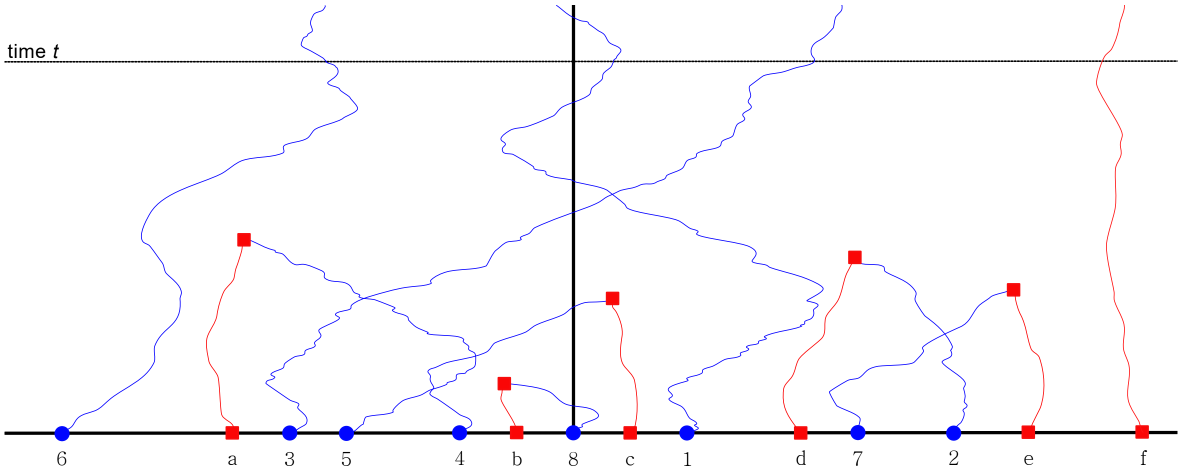


Graphical construction of DLAS

- Each site $x \in G$ is given with discrete SRW path $W^x = (W_k^x)_{k \in \mathbb{N} \cup \{0\}}$ which it follows according to a rate λ_A or λ_B Poisson point process depending on the type of initial particle at x



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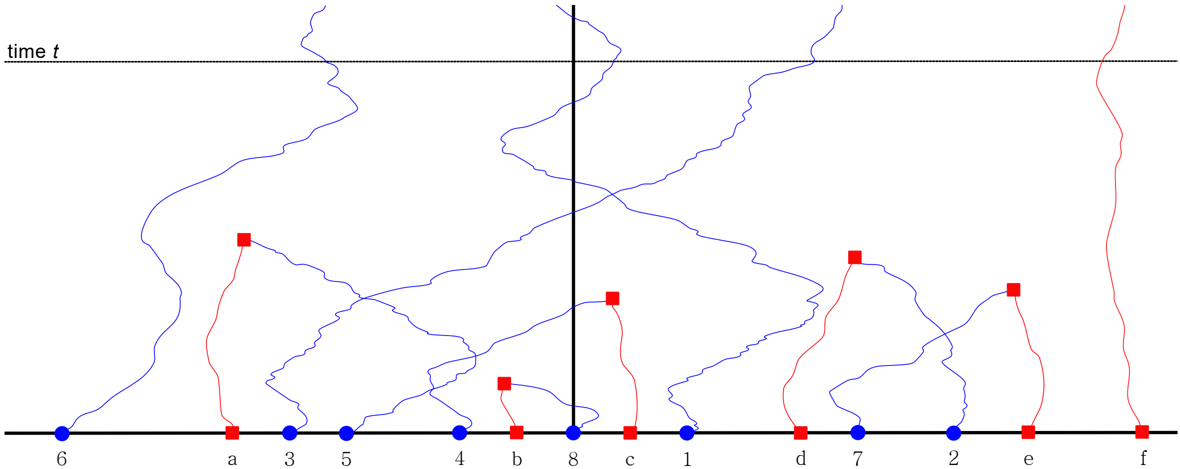

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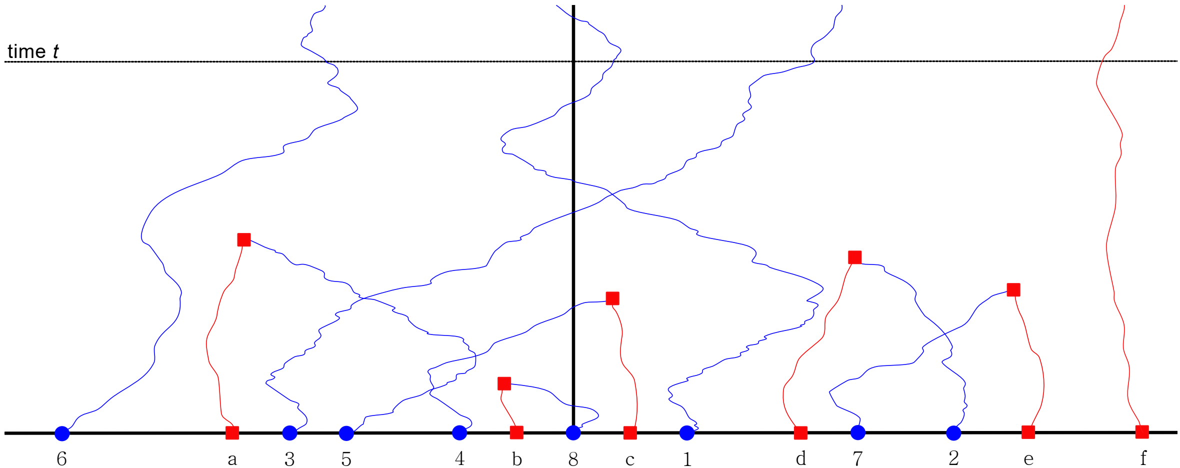

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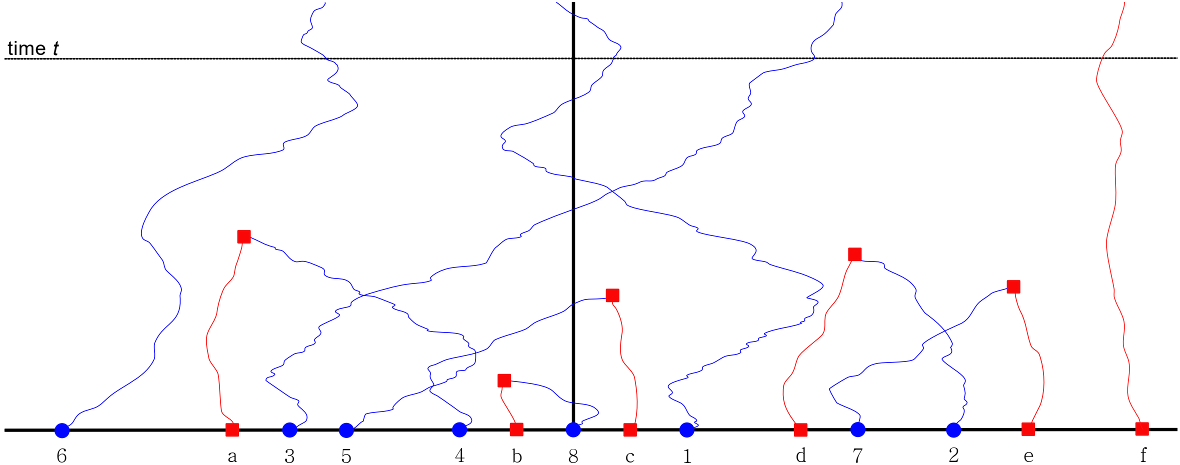

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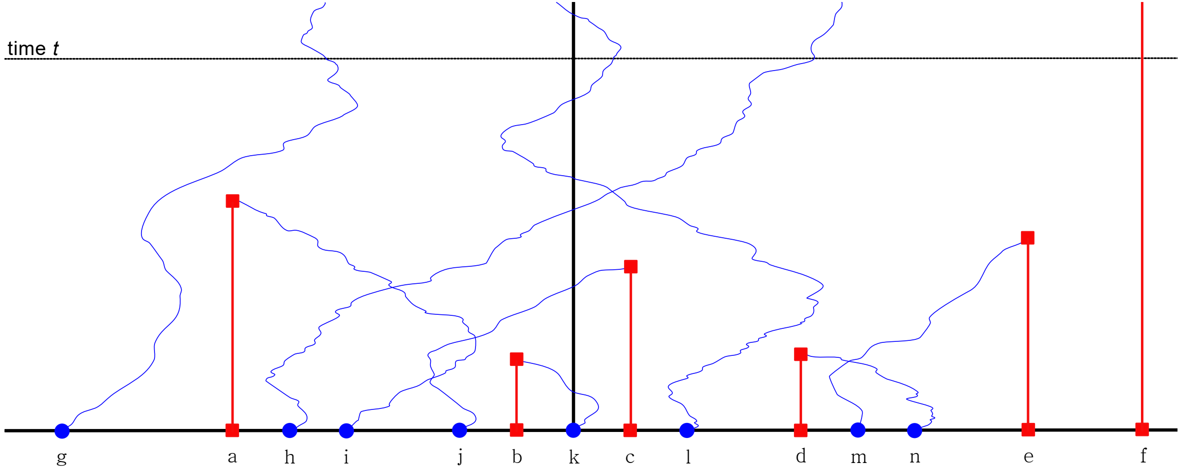

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- Main question:

$$\rho_t \quad \text{or} \quad \mathbb{E}[V_t] \quad \sim \quad f(t; G, p_A, \lambda_A, \lambda_B, \text{Interaction rules})?$$

Historic notes

- Bramson and Lebowitz ([4] '80): For *coalescing RW on* $\mathbb{Z}^d$:

$$\rho_t^{\mathsf{CRW}} \asymp \begin{cases} t^{-1/2}, & d=1 \\ t^{-1}\log t, & d=2 \\ t^{-1}, & d\geq 3 \end{cases}.$$

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- There is no known tractable dual process for DLAS or coupling to well-known processes

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- Physicists have been interested in DLAS as a model for irreversible reactions with mobile particles since 70's [16, 18].
  - Mean-field prediction:

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  - For $p_A < p_B$ and $\lambda_A = \lambda_B$:

$$\log(\rho_t) \asymp -\psi_d(p,t), \qquad \psi_d(p,t) = \begin{cases} \frac{(1-2p)^2}{1-p}\sqrt{t}, & d = 1 \\ (1-2p)\frac{t}{\log t}, & d = 2 \\ (1-2p)\,t, & d \ge 3. \end{cases}$$

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- These asymptotics were also conjectured to hold for different jump rates [15, 14, 13].

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  - So it is unclear whether we should expect the same asymptotics in the symmetric ($\lambda_A = \lambda_B$) and the asymmetric $\lambda_A \neq \lambda_B$) setting

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- Przykucki, Roberts, and Scott ([17] '20): For discrete-time totally asymmetric DLAS on $\mathbb{Z}$ with $p_A = 0.5$,

$$t^{3/4}(\log t)^{-1/4} \lesssim \mathbb{E}[V_t] \lesssim t^{3/4}$$

Historic notes

- Cristali, Jiang, Junge, Kassem, Sivakoff, and York ([8]' 21): Discretized version of DLAS on finite graphs and studied the time to extinguish all particles
- Ahlberg, Griffiths, and Janson ([1] '21) The critical behavior of an two-type annihilating system of branching random walks
- Dauvergne and Sly ([11] '21) Variant of DLAS with reaction $A+B \to A$ (infection spread)
- Bahl, Barnet, Junge, and Johnson ([3] '21): Increasing convex ordering in DLAS

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- Confirming $\rho_t = \tilde{O}(t^{-d/4})$ for $d = 2,3$ is a major open problem (for us:))

Statement of results

Theorem (Critical exponent of occupation time; Johnson, Junge, L., Sivakoff [12] '23)

Let $\lambda_A = 1$ and $\lambda_B \in [0,1]$. On $\mathbb{Z}^d$ for $d \leq 3$ there exists a constant $C > 0$ such that

$$\mathbb{E}_p[V_\infty] \geq C \frac{(1-2p)^{-(4/d)+1}}{-\log(1-2p)}$$

for all $1/4 < p < 1/2$.
For $\lambda_B = 0$ on $\mathbb{Z}$, there exists $C > 0$ such that

$$\mathbb{E}_p[V_\infty] \leq C(1-2p)^{-3}$$

for all $p < 1/2$.

- For $d = 1$ with $\lambda_B = 0$, the above yields

$$\frac{(1-2p)^{-3}}{-\log(1-2p)} \lesssim \mathbb{E}_p[V_\infty] \lesssim (1-2p)^{-3},$$

determining the critical exponent up to logarithmic terms.

Statement of results

Theorem (Mean-field behavior in high dimension; Johnson, Junge, L., Sivakoff [12] '23)

Let $\lambda_A = 1$ and $\lambda_B = 0$. For some positive absolute constants c and C, it holds for all $d \geq 2$ on bidirected $2d$-regular tree with $p = 1/2$ that

$$c \log t \leq \mathbb{E}[V_t] \leq C \log t$$

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For some positive absolute constants c, C, and η, it holds for all $\frac{1}{2} - \eta < p < \frac{1}{2}$ and $d \geq 2$ that

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- Was known for $\mathbb{Z}^d$ for $d \geq 4$ in the symmetric case by Bramson and Lebowitz
- It has not been proven to occur with unequal jump rates on any graph.
  - The lower bound does not follow from that of Cabezas, Rolla, and Sidoravicius ([7] '18)
  - RW kernel on bidirected trees are is not reflective

Sketch of proofs

Lemma (Mass transport principle (MTP))

Let $G=(\mathcal{V},\mathcal{E})$ be the Cayley graph of an infinite group F generated by a finite set S. $Z:\mathcal{V}^2\to[0,\infty)$ be a collection of random variables such that $\mathbb{E}[Z(x,y)]=\mathbb{E}[Z(gx,gy)]$ for all $g\in F$. Let $\mathbb{B}\subseteq\mathcal{V}$ be a subset which is invariant under the inversion $x\mapsto x^{-1}$. Then we have

$$\mathbb{E}\left[\sum_{y\in\mathbb{B},} Z(\mathbf{0},y)\right]=\mathbb{E}\left[\sum_{y\in\mathbb{B}} Z(y,\mathbf{0})\right].$$

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- By the hypothesis, $y\in\mathbb{B}$ if and only if $y^{-1}\in\mathbb{B}$.
- By Fubini's theorem and the invariance of Z in law under diagonal group action,

$$\mathbb{E}\left[\sum_{y\in\mathbb{B}} Z(\mathbf{0},y)\right]=\sum_{y\in\mathbb{B}}\mathbb{E}[Z(\mathbf{0},y)]=\sum_{y\in\mathbb{B}}\mathbb{E}[Z(y^{-1},\mathbf{0})]=\sum_{y\in\mathbb{B}}\mathbb{E}[Z(y,\mathbf{0})]=\mathbb{E}\left[\sum_{y\in\mathbb{B}} Z(y,\mathbf{0})\right].$$

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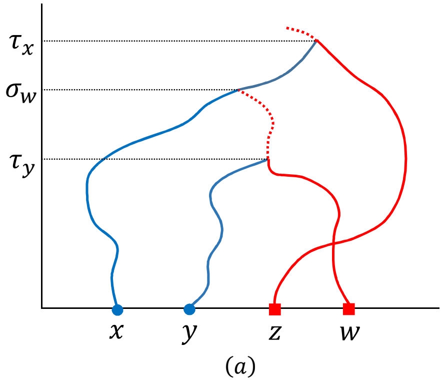

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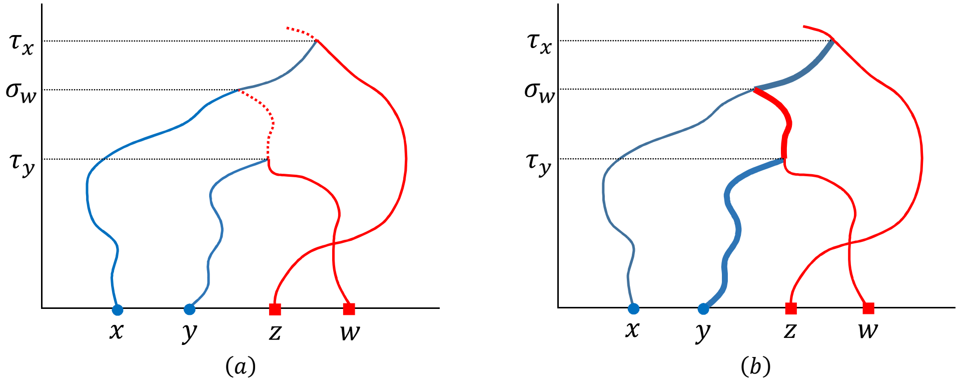

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  - Each token (y) performs a random walk on G, with rate alternating between λ_A and λ_B depending on which type of particle it is being carried

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$$\leq t^{d/2} \exp(-ct)$$

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Let $\lambda_A = 1$ and $\lambda_B = 0$. On $\mathbb{Z}$ with $p = 1/2$, there exists $C > 0$ such that

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  - Label the initial cars on $\mathbb{Z}_{\geq 0}$ from left to right as $1, 2$, and so on.
  - Release car 1 and let it drive until time t. Same $A + B \to \emptyset$ rule applies. If car survives up to time t, kill it at time t.
  - Release car 2 and let it drive until time t in the environment in the previous step. Repeat for all remaining cars sequentially.

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Let $\lambda_A = 1$ and $\lambda_B = 0$. On $\mathbb{Z}$ with $p = 1/2$, there exists $C > 0$ such that

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for all sufficiently large t.

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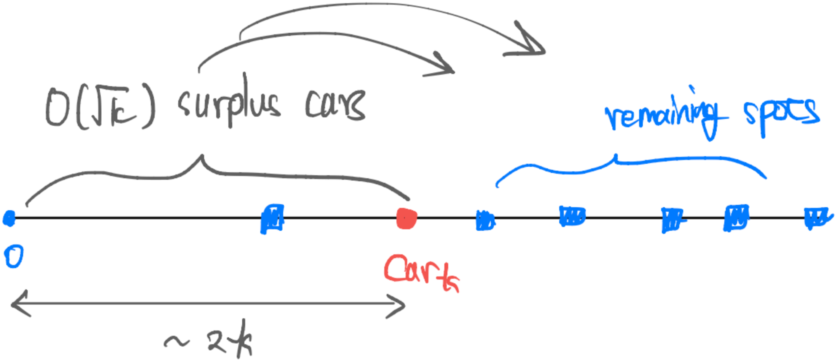

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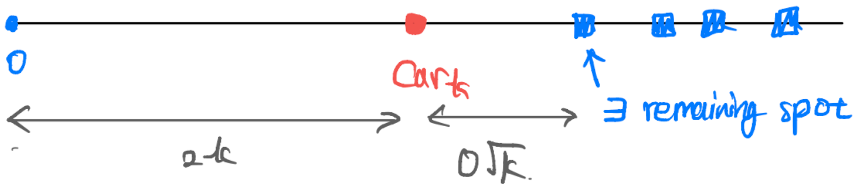

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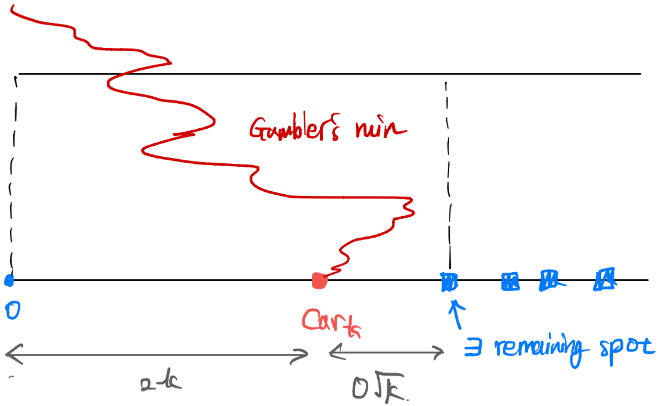

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- Once a particle visits the origin, it spends $O(\sqrt{t})$ time there by time t
- $$\begin{aligned}\mathbb{E}[V_t] &= O\left(\mathbb{E}[V_t^{\ge 0}]\right) \qquad (\because \text{claim 1})\\ &= O\left(\mathbb{E}[V_t^{\text{sequential}}]\right) \qquad (\because \text{claim 2})\\ &= O\left(\sqrt{t}\left(\sum_{k=1}^{C\sqrt{t}\log t} \mathbb{P}\left(k\text{th car in the sequential process visits } \mathbf{0} \text{ by time } t\right) + o(t^{1/4})\right)\right)\\ &= \tilde{O}\left(t^{1/2}\cdot t^{1/4}\right).\end{aligned}$$

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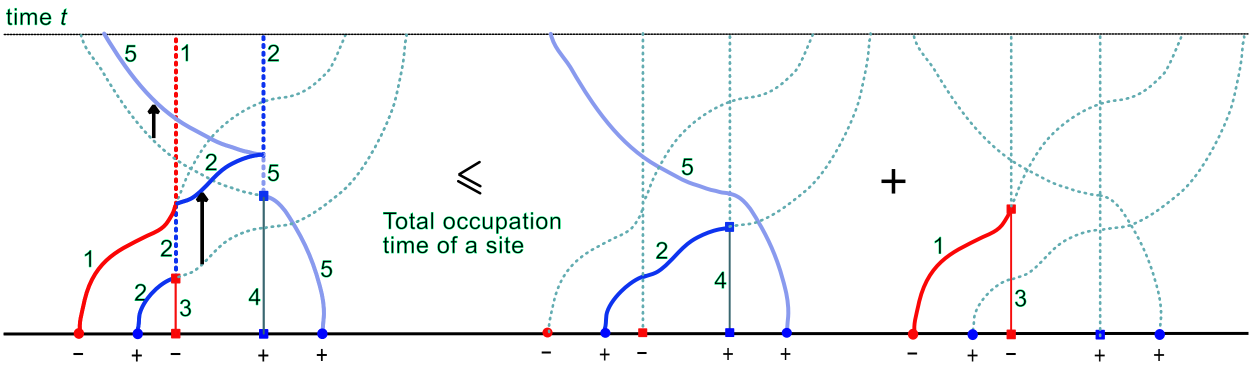

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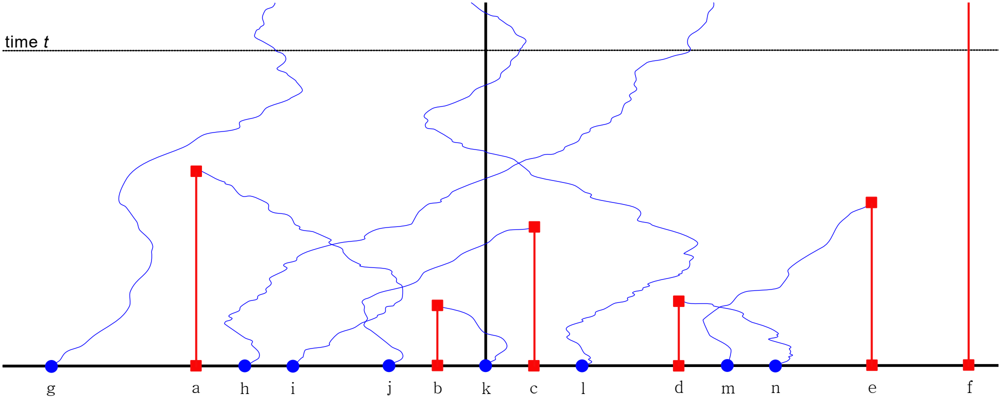

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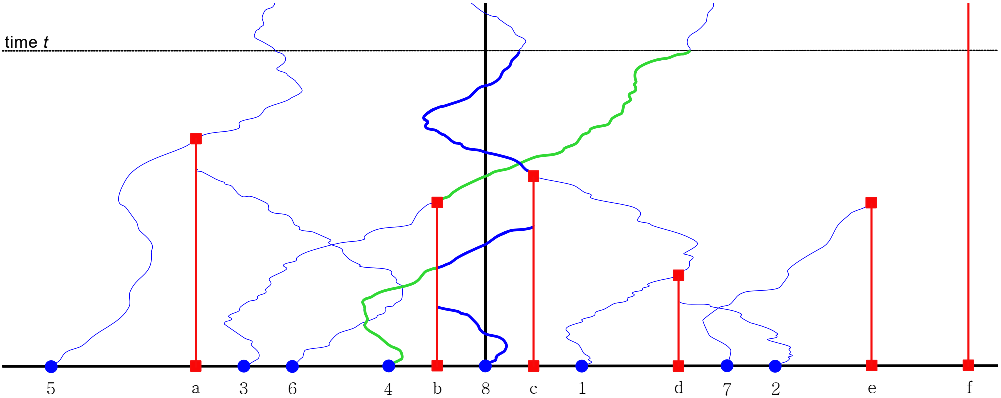
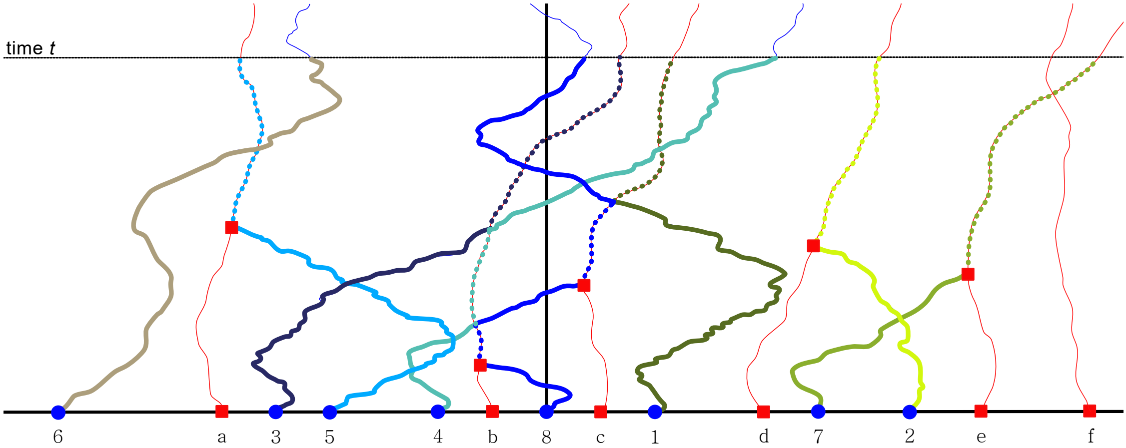


- Remark: Path-swapping DLAS ⇒ 'Quasi-abelian' property of DLAS
  - $\mathbb{E}[V_t] = \mathbb{E}[V_t^{\text{path-swapping}}] = \mathbb{E}[V_t^{\text{seq}}] - \text{dragging}$



Theorem (Mean-field behavior in high dimension)

Let $\lambda_A = 1$ and $\lambda_B = 0$. For some positive absolute constants c and C, it holds for all $d \geq 2$ on bidirected $2d$-regular tree with $p = 1/2$ that

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- Taking $\mathbb{E}$ gives the difference equation:

$$\mathbb{E}W_{n+1} - \mathbb{E}W_n = \frac{1}{2}\mathbb{P}(W_n = 0) \overset{\text{Claim 3}}{\approx} e^{-c\mathbb{E}W_n}.$$

  This yields $\mathbb{E}W_n \asymp \log n$.

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    - Size-biasing does not change W_n much $\Longrightarrow$ W_n is concentrated around $\mathbb{E}[W_n]$

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- Extend the reuslt of Damron, L. and Sivakoff ([9] '22) to continuous time totally asymmetric DLAS with $p_A < 0.5$:

$$\rho_t = \exp(-O(t^{d/(d+2)}))$$

References I

[1] Daniel Ahlberg, Simon Griffiths, and Svante Janson. "To fixate or not to fixate in two-type annihilating branching random walks". In: *Ann. Probab.* 49.5 (2021), pp. 2637–2667. ISSN: 0091-1798. DOI: 10.1214/21-aop1521. URL: https://doi.org/10.1214/21-aop1521.

[2] Richard Arratia. "Limiting point processes for rescalings of coalescing and annihilating random walks on Zd". In: *The Annals of Probability* (1981), pp. 909–936.

[3] Riti Bahl et al. "Diffusion-limited annihilating systems and the increasing convex order". In: *arXiv preprint arxiv:2104.12797* (2021).

[4] Maury Bramson and David Griffeath. "Asymptotics for interacting particle systems on $\mathbb{Z}^d$". English. In: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 53.2 (1980), pp. 183–196. ISSN: 0044-3719. DOI: 10.1007/BF01013315. URL: http://dx.doi.org/10.1007/BF01013315.

[5] Maury Bramson and Joel L. Lebowitz. "Asymptotic behavior of densities for two-particle annihilating random walks". In: *J. Statist. Phys.* 62.1-2 (1991), pp. 297–372. ISSN: 0022-4715. DOI: 10.1007/BF01020872. URL: https://doi.org/10.1007/BF01020872.

References II

[6] Maury Bramson and Joel L. Lebowitz. "Asymptotic Behavior of Densities in Diffusion-Dominated Annihilation Reactions". In: *Phys. Rev. Lett.* 61 (21 1988), pp. 2397–2400. DOI: 10.1103/PhysRevLett.61.2397. URL: https://link.aps.org/doi/10.1103/PhysRevLett.61.2397.

[7] M. Cabezas, L. T. Rolla, and V. Sidoravicius. "Recurrence and density decay for diffusion-limited annihilating systems". In: *Probab. Theory Related Fields* 170.3-4 (2018), pp. 587–615. ISSN: 0178-8051. DOI: 10.1007/s00440-017-0763-3. URL: https://doi.org/10.1007/s00440-017-0763-3.

[8] Irina Cristali et al. "Two-type annihilating systems on the complete and star graph". In: *Stochastic Process. Appl.* 139 (2021), pp. 321–342. ISSN: 0304-4149. DOI: 10.1016/j.spa.2021.05.004. URL: https://doi-org.csi.ezproxy.cuny.edu/10.1016/j.spa.2021.05.004.

[9] Michael Damron, Hanbaek Lyu, and David Sivakoff. "Stretched exponential decay for subcritical parking times on $\mathbb{Z}^d$". In: *Random Structures Algorithms* 59.2 (2021), pp. 143–154. ISSN: 1042-9832. DOI: 10.1002/rsa.21001. URL: https://doi-org.csi.ezproxy.cuny.edu/10.1002/rsa.21001.

References III

[10] Michael Damron et al. "Parking on transitive unimodular graphs". In: *The Annals of Applied Probability* 29.4 (2019), pp. 2089–2113.

[11] Duncan Dauvergne and Allan Sly. "Spread of infections in a heterogeneous moving population". In: *arXiv preprint arXiv:2105.11947* (2021).

[12] Tobias Johnson et al. "Particle density in diffusion-limited annihilating systems". In: *Ann. Probab., to appear* ().

[13] K. Kang and S. Redner. "Scaling Approach for the Kinetics of Recombination Processes". In: *Phys. Rev. Lett.* 52 (12 1984), pp. 955–958. DOI: 10.1103/PhysRevLett.52.955. URL: https://link.aps.org/doi/10.1103/PhysRevLett.52.955.

[14] Zbigniew Koza. "The long-time behavior of initially separated $A + B \to 0$ reaction-diffusion systems with arbitrary diffusion constants". In: *J. Statist. Phys.* 85.1-2 (1996), pp. 179–191. ISSN: 0022-4715. DOI: 10.1007/BF02175561. URL: https://doi-org.csi.ezproxy.cuny.edu/10.1007/BF02175561.

[15] Benjamin P. Lee and John Cardy. "Renormalization group study of the $A + B \to \emptyset$ diffusion-limited reaction". In: *J. Statist. Phys.* 80.5-6 (1995), pp. 971–1007. ISSN: 0022-4715. DOI: 10.1007/BF02179861. URL: https://doi-org.csi.ezproxy.cuny.edu/10.1007/BF02179861.

References IV

[16] A. A. Ovchinnikov and Ya. B. Zeldovich. “Role of density fluctuations in bimolecular reaction kinetics”. In: *Chem. Phys.* 28.1–2 (1978), pp. 215–218.

[17] Michał Przykucki, Alexander Roberts, and Alex Scott. “Parking on the integers”. In: *arXiv preprint arXiv:1907.09437* (2019).

[18] Doug Toussaint and Frank Wilczek. “Particle–antiparticle annihilation in diffusive motion”. In: *J. Chem. Phys.* 78.5 (1983), pp. 2642–2647.

Thank you very much!